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LETTER TO THE EDITOR

Universality along the critical line of the Ising cellular automata

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Abstract. We check the hypothesis of universality along the critical line of the Ising cellular automata (ICA) by measuring critical exponents of damage propagation at the transition temperature T_k in zero field and at higher temperatures in non-zero field and at the Kauffman limit ($T=\infty$). Numerical results support the claim of universality.

The recent formulation of cellular automata equivalent to the Ising model has shown some interesting effects of the introduction of determinism to a thermodynamic system [1, 2]. The Ising cellular automata, or ICA, are spin-$\frac{1}{2}$ automata of the Ising model. In contrast to the Ising model where the response of a site in the system to its local configuration may be different each time that configuration is encountered, in ICA a site's response to a particular local configuration is always the same. The Ising cellular automata evolve according to a set of rules: these rules govern the system's response to any and all possible configurations of neighbouring sites. In particular these rules store whether a site will assume the +1 state or the −1 state in response to a local configuration of neighbouring sites.

Of the utmost importance in ICA is the Boltzmann probability, given by:

$$P(+1)=\frac{\exp(-E(+1)/kT)}{\exp(-E(+1)/kT)+\exp(-E(-1)/kT)} \tag{1}$$

$$E(\sigma_i)-\sum_{\langle ij\rangle} J\sigma_i\sigma_j+H\sigma_i \tag{2}$$

where $\sigma_i=\pm1$ is the state of site i, J is the coupling constant between neighbour sites i and j and H is the applied field. The rules determine the subsequent state of a site according to the Boltzmann probability, $P(+1)$, of the site being +1. At the beginning of a simulation, for each site and for each possible local configuration around that site, a random number is compared to the appropriate Boltzmann probability for that configuration and if this random number is less than $P(+1)$ then the rule will be that the site assumes the +1 state, otherwise the rule will be that the site assumes the −1 state. The rule is now fixed for that site and for that particular configuration. To make the distinction between an equilibrium Ising model and ICA clearer, we outline the steps in updating a single site. In the Ising model one would note the local configuration of neighbours, calculate $P(+1)$ and generate a random number. If that random number were less than $P(+1)$ then the site would assume the +1 state, alternatively if the

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random number were greater than $P(+1)$ then the site would be set as -1. The next time that site was updated a new random number would be used in the process. In an ICA one would note the state of the neighbours and their local arrangement. This arrangement uniquely determines the subsequent state of the target binary variable (±1) because the rules, created before any updating had started, already contain the response to that arrangement of neighbours. The next time we update this site we will use the same rules, hence if the local configuration is the same, then the response will be the same.

As a basis for discussion we limit ourselves to simultations and results from the square lattice. In this 2D nearest-neighbour version of ICA a site has 16 rules. We stress that these rules are created before a simulation begins, and no further random numbers are used thereafter. Given the rules that each site will follow and each site's initial state, the subsequent behaviour of the entire system is determined. The ICA is therefore a deterministic, but non-reversible system.

The ICA incorporate a widely known cellular automata, the Kauffman model, as a limiting case. The Kauffman model can be referred to as ICA at high temperature with a field, where the field is related to the probability, p, of the Kauffman model by:

$$H/kT = h = \frac{1}{2} * \ln\left(\frac{1-p}{p}\right). \tag{3}$$

We define $h = H/kT$, which we will refer to as the reduced field, and work with this quantity instead of H. One will notice that the ICA offer a fundamental and meaningful method of including temperature in the Kauffman model [3].

Perhaps the most striking and intriguing property of the ICA in 2D may be found in comparison of damage propagation of the ICA with that of the equilibrium Ising model subject to heat-bath dynamics. Damage may be viewed as the deviation or Hamming distance of two initially identical systems (same rules, same initial configuration) after a small perturbation to one system. During the temporal evolution of the systems, if coinciding sites in the two systems have different states, then we refer to the site as damaged. The collection of damaged sites is known as the damage cloud. When a small initial defect causes the damage to propagate to the edge of a system of any size, we say the system is in the chaotic phase. When this initial damage remains localized we refer to the system as being in a frozen phase. In our simulations of the ICA, the initial damage is a distinct set of rules governing the evolution of a row of spins at the centre of the second system which are different from those in the first system. The rules governing the behaviour of the other sites are identical for both systems. This method of introducing initial damage has been shown to be equivalent to flipping the centre line of spins in a second system, this second system being an exact replica of the first [4].

Figure 1 shows the global dynamic phase diagram of both systems; the 2D Ising model and 2D ICA. The traditional Ising model has a singularity at $T = T_c = 2/\ln(\sqrt{2}+1)$, $h = 0$, at which the damage will propagate to the edge of an infinite system [5]. The ICA has a phase separation, with a line of critical points. Above this line is the frozen regime and below is the chaotic regime. Much of the interest lies along the boundary between these two regimes. Does the system evolve as in the well studied limiting case, the Kauffman model, along the entire critical line, or do the critical properties of the system vary continuously along this line? Some recent work [6], along with the work presented here support the former proposition; namely that the exponents are constant along the critical surface.

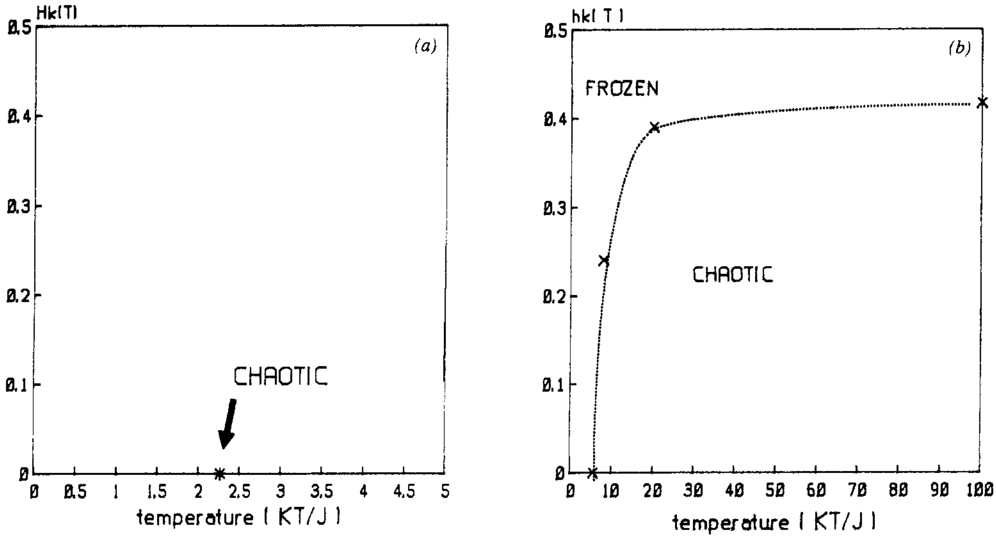

Figure 1. (a) Dynamic global phase diagram for Ising model. Note the singularity at T_c, at which damage will propagate to the edge of an infinite system. $T_c \approx 2.269$. (b) Dynamic global phase diagram for ICA. Note the critical line between the frozen (damage localized) and the chaotic (damage propagates to edge of infinite system) phases. When $h = 0$, the critical temperature is $T_k \approx 5.6$.

One may describe the spread of damage through the system using the critical exponents d_{act} and d_t, introduced by Stauffer for the Kauffman model [7]:

$$\langle m \rangle \sim L^{d_{\text{act}}} \tag{4}$$

$$\langle t \rangle \sim L^{d_t} \tag{5}$$

where M is the mass or number of damaged sites in the damage cloud when the damage reaches the edge of the system and t is the time for the damage to touch the edge. The critical exponents for the ICA in zero field at $T = T_k \approx 5.6$ were quoted by Jan as $1.8 < d_{\text{act}} < 1.9$ and $d_t \approx 1.3$ [6]. These values are in agreement with those found for the Kauffman model or ICA at $T = \infty$ and $h = h_k(T)$ ($h_k(T)$ is the critical reduced field at temperature T), clarified recently by Stauffer [8]. Jan proposed that these exponents are the same all along the critical line, implying that these ICA are members of the same universality class. Results of the work presented here agree with this conjecture.

Simulations were done to find the critical reduced field, $h_k(T)$, at suitable temperatures, where $h_k(T)$ is the reduced field necessary to prevent damage spreading to the edge of an infinite system in at least 50% of the simulations at temperature T. We determine h_k in a manner similar to that of Stauffer [8] in his odyssey to find p_c for the Kauffman model. The variation of h_k with $1/L$ was plotted, and the asymptotic value as L tends to infinity was noted. At $T = 8.0$, the reduced critical field h_k is 0.24. While at $T = 20.0$, a very high temperature, h_k is 0.39. The reduced critical field found at $T = \infty$ (from the most reliable estimate of p_c Kauffman) is $h_k = 0.4236$. The critical exponents, defined in equations 4 and 5, may be found by graphing $\log(M)$ and $\log(t)$ against the $\log(L)$. The critical exponents for $T = T_k = 5.6$, $T = 8.0$ and $T = 20.0$ are found from figure 2. Table 1 lists the numerical values of these exponents along with

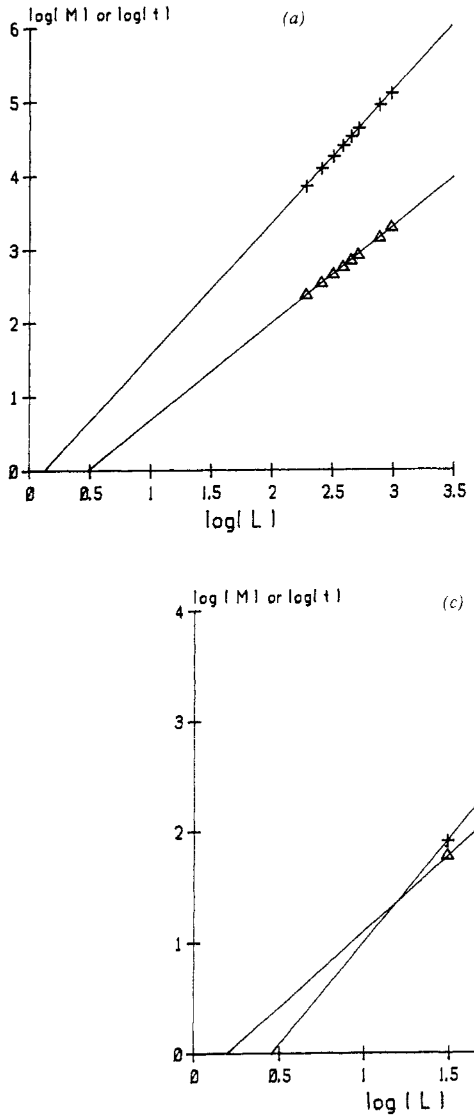

Figure 2. Critical exponents: $\log(\langle M\rangle)$ and $\log(\langle t\rangle)$ against $\log(L)$. Where M is the damage at time t, when the damage reaches the edge of a system of size L. The slopes of the lines will yield the critical exponents d_{act} and d_t as the scaling relations are $\langle M\rangle \sim L^{d_{\text{act}}}$ and $\langle t\rangle \sim L^{d_t}$. (*a*) $T = T_k = 5.6$, $d_{\text{act}} = 1.81 \pm 0.05$, $d_t = 1.31 \pm 0.05$; (*b*) $T = 8.0$, $d_{\text{act}} = 1.85 \pm 0.05$, $d_t = 1.34 \pm 0.05$; (*c*) $T = 20.0$, $d_{\text{act}} = 1.84 \pm 0.05$, $d_t = 1.36 \pm 0.05$. + = damage, $\triangle$ = time.

Table 1.

ICA	$T = T_k = 5.6$	$T = 8.0$	$T = 20.0$	$t = \infty$ Kauffman [8]	Ising model [9]
d_{act}	1.81 ± 0.05	1.85 ± 0.05	1.84 ± 0.05	1.85 ± 0.05	1.72 ± 0.03
d_t	1.31 ± 0.05	1.34 ± 0.05	1.36 ± 0.05	1.35 ± 0.05	2.24 ± 0.04

those for the Kauffman model or ICA at $T = \infty$, and the equilibrium Ising model. Comparison of these values shows that along the critical surface of the ICA, the critical exponents are numerically indistinguishable within the quoted error bars.

To summarize we have explored the dynamics of the newly formulated ICA along the critical surface. We have found that numerical results support the hypothesis of universality along this critical line: the relevant exponents $d_{act} \approx 1.85 \pm 0.5$ and $d_t \approx 1.3 \pm 0.5$ are within quoted errors, constant along the critical line. Further these results combined with results of work on the Ising model [9] show that the ICA lie in a distinct universality class from the equilibrium Ising model.

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